Lecture 2

Getting Started

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We seek a theory of bidding behavior in auctions.

Our theory will attempt to explain how peoples' bids are related to their individual valuations, or simply *values*, for the item being auctioned.

In mathematical terminology, we want a mapping from values to bids.

A person's value is the hypothetical price at which she is indifferent between buying the item and not buying it.

With this interpretation, the payoff someone gets from winning an item in an auction is the difference between her value and the actual price she pays for it.

We cannot presume to know peoples' values for items they bid on in auctions.

An individual's willingness to pay for something depends on...

- how much satisfaction or enjoyment she gets from consuming it instead of other things she could potentially consume in its place
- their wealth or disposable income

These attributes are (typically) unobservable.

We can't proceed unless we know something about individual values.

We assume that we know the range of possible values that people hold and the likelihood that someone's value lies in any interval.

Probability distribution function, F

Let v_i denote individual i's value.

From the point of view of the auctioneer, or other bidders, or an economist analyzing the auction, individual i's value is a random variable $\tilde{v}_i$.

The information the auctioneer has about the likelihood of different realizations of the random variable $\tilde{v}_i$ is summarized by the function F.

For any value v, the number $F(v)$ tells us the probability that an individual has a value that is less than or equal to v.

Interpretation

You can think of the function F as representing the distribution of types of people that exist in the population.

$F(v)$ is the fraction of the population with types less than v.

An individual who shows up at an auction is a random draw from the population.

Hence, the likelihood that she has a value in any range is simply given by the fraction of the population that has values in that range.

Assumptions

It is convenient to assume that the lower bound on peoples' values is 0.

When values range from 0 to some maximum value $v_{\max}$, we require that $F(0) = 0$ and $F(v_{\max}) = 1$.

A convenient special case arises when $F(v) = v/v_{\max}$ for any value v between 0 and $v_{\max}$.

Then we say values are *uniformly distributed* on $[0, v_{\max}]$.

This means all value types between 0 and $v_{\max}$ are equally represented in the population, and hence the probability of seeing an individual at an auction with a value in the first one-third of the value range is $1/3$, in the first half of the value range is $1/2$, etc.

Probability value lies in an interval

The probability of observing a value in any interval is simply equal to the ratio of the length of that interval and the length of the value range.

For instance, if the value range is 0 to \$100, the probability of drawing a value between \$30 and \$70 is $(70 - 30)/100 = 0.4$.

This is just the difference between the probabilities associated with the upper and lower bounds of the interval.

Summary

The preceding description of values is formalized as follows:

- Bidder i's value is denoted v_i and is assumed to lie in the interval $[0, v_{\max}]$.
- $\tilde{v}_i$ is a random variable whose realization v_i is the use value of bidder i.
- $F(\cdot)$ is the probability distribution function of $\tilde{v}_i$; $F(v) = \Pr[\tilde{v}_i \leq v]$.

We make the following assumptions about the probability distribution function F.

Assumption 1. $F(\cdot)$ is a non-decreasing function on $[0, v_{\max}]$.

Assumption 2. $F(0) = 0$ and $F(v_{\max}) = 1$.

Assumption 3. $F(\cdot)$ has a continuous and positive derivative on $[0, v_{\max}]$, $f(v) = F'(v)$.

Assumptions 1 and 2 are required assumptions for a probability distribution function. Assumption 3 is for convenience. We call $f(\cdot)$ the *density function* of the random variable $\tilde{v}_i$. Note that since $F(\cdot)$ has a continuous and positive derivative on $[0, v_{\max}]$, no individual value in the interval $[0, v_{\max}]$ occurs with strictly positive probability, and hence

$$\Pr[\tilde{v}_i < v] = \Pr[\tilde{v}_i \leq v] = F(v).$$

This means we do not have to distinguish between outcomes that are "less than" some number and outcomes that are "less than or equal to" some number.

Order Statistics

Our predictions for auction outcomes will depend on what we expect bidders' values to be.

We don't need to predict all bidders' values.

What usually matters are the highest or second-highest values. These are what typically determine who wins an auction or how much they pay.

In an auction with n bidders, the highest and second-highest values are called the nth and $(n-1)$th order statistics, respectively.

Since the vector of values that bidders possess is not known to us, we must treat each of these order statistics as a random variable.

Given our knowledge of the probability distribution function F for individual values we can compute probability distribution functions for each of these random variables.

Computing $F_{(1)}$

Let $\tilde{v}_{(1)}$ and $\tilde{v}_{(2)}$ denote the random variables that correspond to the highest and second-highest elements of $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$, the set of random variables that represent all the individual values.

We want the probability distribution functions for $\tilde{v}_{(1)}$ and $\tilde{v}_{(2)}$.

First consider the random variable $\tilde{v}_{(1)}$.

In order for $\tilde{v}_{(1)}$ to be less than v, it must be the case that each of the individual values is less than v.

That is, we require $\tilde{v}_1 < v$, $\tilde{v}_2 < v$, $\tilde{v}_3 < v$, etc., all the way up to $\tilde{v}_n < v$.

The probability that any $\tilde{v}_i < v$ is $\dfrac{v}{v_{\max}}$ — that is simply the definition of $F(v)$.

Since we assume the n bidders' values are independently distributed, the probability that the highest of the n random variables, $\tilde{v}_1$ through $\tilde{v}_n$, is less than or equal to v is the product of the n probabilities that each one of them is less than v:

$$F_{(1)}(v) = \Pr[\tilde{v}_{(1)} \leq v] = \prod_{i=1}^{n} \Pr[\tilde{v}_i \leq v] = \left(\frac{v}{v_{\max}}\right)^n.$$

Example

Suppose the random variable $\tilde{v}_i$ is uniformly distributed on the interval $[0, 100]$. Then

$$F(v) = \frac{v}{100}$$

$$F_{(1)}(v) = \left(\frac{v}{100}\right)^n$$

$$f_{(1)}(v) = F'_{(1)}(v) = \frac{nv^{n-1}}{100^n}$$

Computing $F_{(2)}$

Now consider the random variable $\tilde{v}_{(2)}$.

There are a number of ways that the realization of the random variable $\tilde{v}_{(2)}$ can be less than v.

We already considered one of them:

- each of the individual values is less than v
- this occurs with probability $\left(\frac{v}{v_{\max}}\right)^n$

Another way for the realization of $\tilde{v}_{(2)}$ to be less than v is for $\tilde{v}_1$ to be greater than v and for each of the other random variables, $\tilde{v}_2$ through $\tilde{v}_n$, to be less than v.

The probability that $\tilde{v}_1$ is greater than v is $1 - F(v) = 1 - \frac{v}{v_{\max}}$.

The probability that all of the other $n - 1$ random variables are less than v is $\left(\frac{v}{v_{\max}}\right)^{n-1}$.

Combined, the probability that $\tilde{v}_1$ is greater than v and all the rest are less than v is

$$\left(1 - \frac{v}{v_{\max}}\right)\left(\frac{v}{v_{\max}}\right)^{n-1}.$$

This can happen n different ways.

The probability that (any) one value is greater than v and all the rest are less than v is

$$n\left(1-\frac{v}{v_{\max}}\right)\left(\frac{v}{v_{\max}}\right)^{n-1}.$$

There is no other way for the second-highest value to be less than v.

Hence, the probability that the second-highest of the n random variables $\tilde{v}_i$, $i=1,\ldots,n$, is less than or equal to v is

$$F_{(2)}(v)=\Pr[\tilde{v}_{(2)}\leq v]=\left(\frac{v}{v_{\max}}\right)^{n}+n\left(1-\frac{v}{v_{\max}}\right)\left(\frac{v}{v_{\max}}\right)^{n-1}.$$

Example

Suppose the random variable $\tilde{v}_i$ is uniformly distributed on the interval $[0, 100]$. Then

$$
\begin{aligned}
F_{(2)}(v) &= \left(\frac{v}{100}\right)^n + n\left(1 - \frac{v}{100}\right)\left(\frac{v}{100}\right)^{n-1} \\
&= \left(\frac{v}{100}\right)^n + n\left[\left(\frac{v}{100}\right)^{n-1} - \left(\frac{v}{100}\right)^n\right] \\
&= n\left(\frac{v}{100}\right)^{n-1} - (n-1)\left(\frac{v}{100}\right)^n \\
f_{(2)}(v) = F'_{(2)}(v) &= n(n-1)\left(\frac{v^{n-2}}{100^{n-1}} - \frac{v^{n-1}}{100^n}\right).
\end{aligned}
$$

In addition, we will sometimes use the random variable

$$\tilde{y} = \max\{\tilde{v}_1, \dots, \tilde{v}_{i-1}, \tilde{v}_{i+1}, \dots, \tilde{v}_n\}.$$

This is the maximum of the values of all the bidders excluding bidder i.

Since we assume that all bidders have values determined by the same uniform distribution F, bidder i can be any one of the bidders (including bidder 1 or bidder n).

Let $G(\cdot)$ denote the distribution function for the random variable $\tilde{y}$:

$$G(y) = \left(\frac{y}{100}\right)^{n-1}$$

$$g(y) = G'(y) = \frac{(n-1)\, y^{n-2}}{100^{n-1}}.$$